Accuracy of the numerical inversion of irrational and transcendental Laplace transform using Haar wavelet operational matrix

Zdravko Stanimirović, Ivanka Stanimirović, Slobodanka Galović, Katarina Đorđević, Edin Suljovrujić
Vinča Institute of Nuclear Sciences, National Institute of the Republic of Serbia, University of Belgrade
Belgrade, Serbia
zdravko.stanimirovic@vinca.rs; ivanka.stanimirovic@vinca.rs; bobagal@vinca.rs; katarina.djordjevic@vinca.rs; edin@vinca.rs

Abstract—**Irrational and transcendental functions can often be seen in signal processing or physical phenomena analysis as consequences of fractional-order and distributed-order models that result in fractional or partial differential equations. In cases when finding solution in analytical form tends to be difficult or impossible, numerical calculations such as Haar wavelet operational matrix method can be used. In order to evaluate accuracy of the numerical inversion of irrational and transcendental Laplace transform using Haar wavelet operational matrix, a number inverse Laplace transforms are numerically solved and compared with the analytical solutions and solutions provided by Invlap and NILT algorithms.**

Key words-Haar wavelet; Laplace transform; Invlap; NILT

I. INTRODUCTION

In physical phenomena analysis and signal processing irrational and transcendental functions can often be seen. They come as consequences of fractional-order and distributed-order models that result in fractional or partial differential equations [1-4]. In cases when finding solution in analytical form tends to be difficult or impossible [5-10], numerical calculations such as Haar wavelet operational matrix [11-14] method can be used. Haar wavelet establishes a direct procedure for transfer function inversion using the wavelet operational matrix for orthogonal function set integration.

In this paper accuracy of the Haar wavelet operational matrix for finding inverse Laplace transforms will be evaluated by comparison of results obtained by Haar wavelet method, analytical results and results obtained by Invlap algorithm and Zakian method based NILT algorithm. Invlap numerical inverse Laplace transform algorithm is based on De Hoog's algorithm [15], while Zakian method based NILT algorithm is based on the Fourier series method with Padé approximation [16]. These two numerical inverse Laplace transform algorithms were chosen because they are effective and can deal with irrational and transcendental functions.

II. HAAR WAVELETS

Haar functions are defined in the interval of $[0,\tau)$ [11-12] by the scaling function $h_0(t)$:

$$h_0(t) = m^{-\frac{1}{2}} \tag{1}$$

and the fundamental square wave $h_1(t)$:

$$h_1(t) = m^{-\frac{1}{2}}\begin{cases} 1, 2^{-j}\tau(k-1) \le t < 2^{-j}\tau(k-1/2) \\ -1, 2^{-j}\tau(k-1/2) \le t < 2^{-j}\tau k \\ 0, \text{elsewere in } [0,\tau) \end{cases} \tag{2}$$

Other wavelets are:

$$h_i(t) = m^{-\frac{1}{2}}\begin{cases} 2^{j/2}, 2^{-j}\tau(k-1) \le t < 2^{-j}\tau(k-1/2) \\ -2^{j/2}, 2^{-j}\tau(k-1/2) \le t < 2^{-j}\tau k \\ 0, \text{otherwise in } [0,\tau) \end{cases} \tag{3}$$

where m is being denoted as the maximum level of resolution and i= 0,1,2…,(m-1), $m=2\alpha$, $\alpha\epsilon Z^+$. Integer decomposition of the index i is designated by j and k.

Any function $x(t)$ can be expanded into a Haar series and in the matrix form it can be given as:

$$\boldsymbol{x}^T = \boldsymbol{c}^T \cdot \boldsymbol{H} \tag{4}$$

where $\boldsymbol{c}^T$ and $\boldsymbol{H}$ are the Haar coefficient vector and the Haar function vector, respectively:

$$\boldsymbol{c}^T = [c_0 \quad c_1 \quad \cdots \quad c_{m-1}] \tag{5}$$

$$\boldsymbol{H} = [h_0 \quad h_1 \quad \cdots \quad h_{m-1}]^T \tag{6}$$

Integration of the Haar wavelet function $\boldsymbol{H}$ can be written as:

$$\int_0^i H(t)dt = \boldsymbol{H} \cdot \boldsymbol{Q_m} \tag{7}$$

where $\boldsymbol{Q_H}$ is the Haar operational matrix for integration:

$$Q_m = (2\cdot m)^{-1}\begin{bmatrix} 2\cdot m\cdot Q_{m/2} & -\tau\cdot H_{m/2}^T \\ \tau\cdot H_{m/2}^T & \mathbf{0}_{m/2}\end{bmatrix}, 0\le t<\tau \quad (8)$$

Therefore, generalized Haar operational matrix can be expressed as:

$$Q_H = H\cdot Q_m\cdot H^T \quad (9)$$

In that case, the inversion of the Laplace transform *X(s)* can be given by:

$$x^T = c^T\cdot H = [2m \quad -2m \quad \dots \quad -2m]_{1\times m}\cdot H^T\cdot \hat{X}(Q_H)\cdot H \quad (10)$$

where $\hat{X}(Q_H)$ is the discrete form of the transfer function *X(s)*.

III. ACCURACY OF THE HAAR WAVELET METHOD – PERIODIC FUNCTION

Let us take the following transfer function into consideration:

$$X(s) = \frac{1}{s\sqrt{s}}\cdot e^{-\frac{k}{s}} \quad (11)$$

When we replace *1/s* by Haar wavelet operational matrix Q_H:

$$\hat{X}(Q_H) = Q_H^{1.5}\cdot e^{-kQ_H} \quad (12)$$

Then the inversion of Laplace transform can be calculated by:

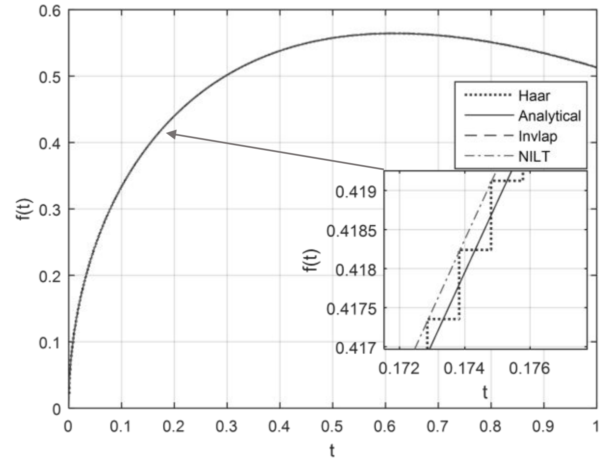
$$x^T = [2m \quad -2m \quad \dots \quad -2m]_{1\times m}\cdot H^T\cdot Q_H^{1.5}\cdot e^{-kQ_H}\cdot H \quad (13)$$

The analytical inverse Laplace transform of the equation (11) is:

$$x(t) = \frac{1}{\sqrt{\pi k}}\cdot \sin\left(2\sqrt{kt}\right) \quad (14)$$

This example is chosen because sinusoidal functions appear everywhere, and they play an important role in circuit analysis. Apart from electrical engineering they are seen in various branches of science and engineering.

In case of k=1, for transfer function *X(s)* given by the equation (11), analytical result and numerical results obtained by Haar wavelet method (m=1024), Invlap and NILT algorithms are shown in Fig. 1(a) for interval [0,1) and in Fig. 1(c) for expanded interval [0,τ). Standard and absolute errors for Haar wavelet method with three different maximum resolution levels (m=64, 256 and 1024) as well as for Invlap and NILT algorithms are presented in Fig. 1(b) for interval [0,1) and in Fig. 1(d) for expanded interval [0,τ).



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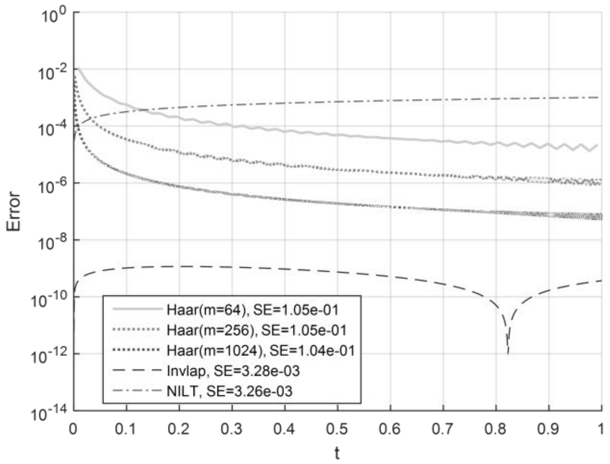

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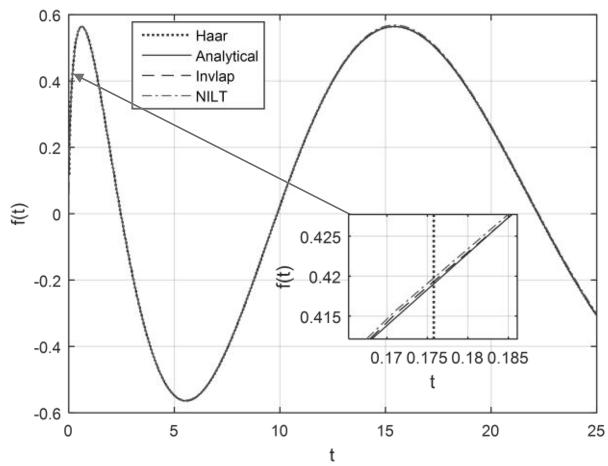

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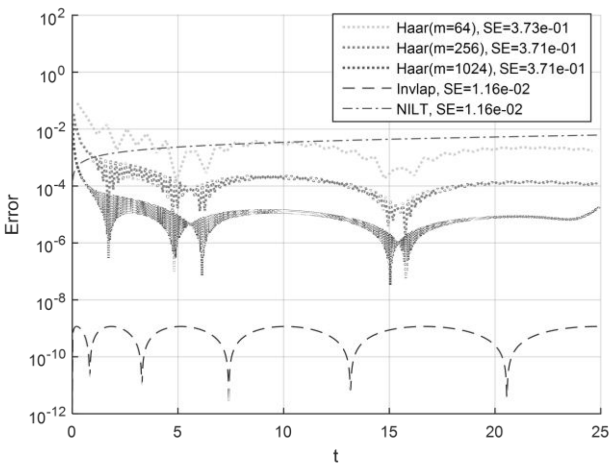

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Fig. 1. The inverse Laplace transform of transfer function *X(s)* given by the equation (11) obtained analytically and numerically by Haar wavelet method (m=1024), Invlap and NILT algorithms for (a) interval [0,1) and (c) expanded interval [0,τ). Standard and absolute errors for Haar wavelet method with three different maximum resolution levels and Invlap and NILT algorithms for intervals (b) [0,1) and (d) [0,τ).

For the transfer function given by the equation (11), for both intervals, Haar wavelet method is in a good agreement with analytical solution as well as other two algorithms. When Haar wavelet method is in question, absolute error is in the 10^{-4}-10^{-6} range for maximum m value during the [0,τ) interval. Haar wavelet method performs better than NILT whose absolute error is almost constant over the entire time span and is of the order of 10^{-2}. Absolute error of Invlap algorithm is of the order of 10^{-10}. Haar wavelet method standard error is of the order 10^{-1} for both intervals and for all three maximum resolution levels. Standard error of Invlap and NILT are of order of 10^{-3} for interval [0,1) and 10^{-2} for the expanded interval [0,τ).

IV. Accuracy of the Haar Wavelet Method – Error Function

Consider the following transfer function:

$$X(s) = \frac{1}{\sqrt{s} \cdot \left(\sqrt{s} + a\right)} \tag{15}$$

In terms of *1/s* transfer function becomes:

$$\hat{X}\left(\frac{1}{s}\right) = \frac{1}{s} \cdot \frac{1}{\left(1 + \frac{a}{\sqrt{s}}\right)} \tag{16}$$

Each *1/s* is then replaced by Haar wavelet operational matrix $\boldsymbol{Q_H}$:

$$\hat{X}(\boldsymbol{Q_H}) = \boldsymbol{Q_H} \cdot (\boldsymbol{I} + \boldsymbol{Q_H^{0.5}} \cdot a)^{-1} \tag{17}$$

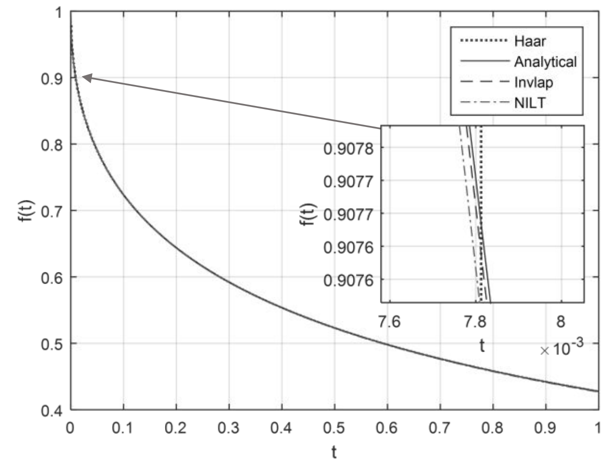
Then, the inversion of Laplace transform can be calculated by:

$$\boldsymbol{x^T} = [2m \quad -2m \quad \ldots \quad -2m]_{1\times m} \cdot \boldsymbol{H}^T \cdot \boldsymbol{Q_H} \cdot (\boldsymbol{I} + \boldsymbol{Q}_H^{0.5} \cdot a)^{-1} \cdot \boldsymbol{H} \tag{18}$$

The analytical inverse Laplace transform of the equation (15) is:

$$x(t) = e^{a^2 t} \cdot erfc\left(a\sqrt{t}\right) \tag{19}$$

In case of a=1, for transfer function *X(s)* given by the equation (15), analytical result and numerical results obtained by Haar wavelet method (m=1024), Invlap and NILT algorithms are shown in Fig. 2(a) for interval [0,1) and in Fig. 2(c) for expanded interval [0,τ). Standard and absolute errors for Haar wavelet method with three different maximum resolution levels (m=64, 256 and 1024) as well as for Invlap and NILT algorithms are presented in Fig. 2(b) for interval [0,1) and in Fig. 2(d) for expanded interval [0,τ).



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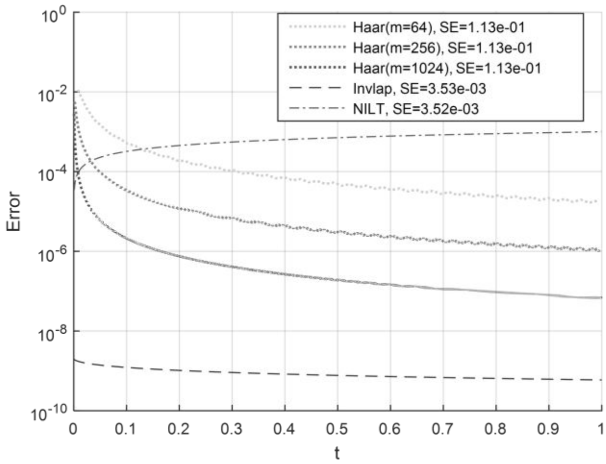

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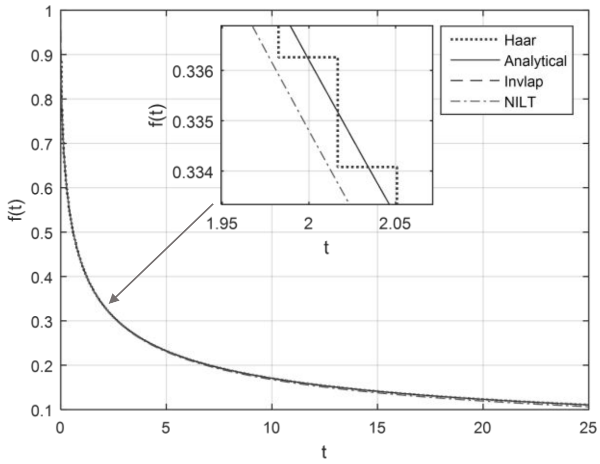

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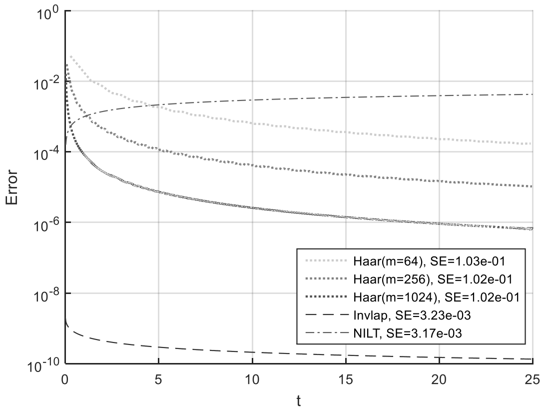

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Fig. 2. The inverse Laplace transform of transfer function *X(s)* given by the equation (15) obtained analytically and numerically by Haar wavelet method (m=1024), Invlap and NILT algorithms for (a) interval [0,1) and (c) expanded interval $[0,\tau)$. Standard and absolute errors for Haar wavelet method with three different maximum resolution levels and Invlap and NILT algorithms for intervals (b) [0,1) and (d) $[0,\tau)$.

For the transfer function given by the equation (15), for both intervals, Haar wavelet method shows a good agreement with analytical solution as well as Invlap and NILT. When Haar wavelet method is in question, absolute error decreases with time over the entire time span. It is in 10^{-2} – 10^{-6} range for maximum m. Standard error is of the order 10^{-1} for all three maximum resolution levels during the whole interval while Invlap and NILT perform better with standard errors of order of 10^{-3}. Absolute error of NILT and Invlap algorithms are almost constant during the whole time. Haar wavelet method performs better than NILT whose absolute error is of the order of 10^{-2}. Absolute error of Invlap algorithm is of the order of 10^{-9}.

V. Accuracy of the Haar Wavelet Method – Error Function and Complementary Error Function

When we take into consideration the following transfer function:

$$X(s) = \frac{b^2 - a^2}{(s - a^2)\left(\sqrt{s} + b\right)} \tag{20}$$

Then, in terms of *1/s,* it becomes:

$$\hat{X}\left(\frac{1}{s}\right) = \frac{1}{s\sqrt{s}} \frac{b^2 - a^2}{\left(1 - \frac{a^2}{s}\right)\left(1 + \frac{b}{\sqrt{s}}\right)} \tag{21}$$

Each *1/s* is then replaced by Haar wavelet operational matrix $\boldsymbol{Q_H}$:

$$\hat{X}(\boldsymbol{Q_H}) = (b^2 - a^2)\boldsymbol{Q}_H^{1.5}[(\boldsymbol{I} - a^2\boldsymbol{Q_H}) \cdot (\boldsymbol{I} + b\boldsymbol{Q_H^{0.5}})]^{-1} \tag{22}$$

Then the inversion of Laplace transform can be calculated by:

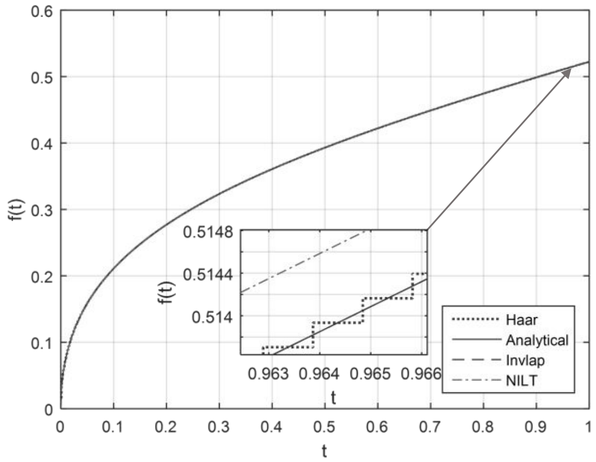
$$\begin{aligned} \boldsymbol{x^T} = [2m \quad -2m \quad \dots \quad -2m]_{1\times m} \cdot \boldsymbol{H}^T \\ \cdot (b^2 - a^2) \cdot \boldsymbol{Q}_H^{1.5} \\ \cdot [(\boldsymbol{I} - a^2\boldsymbol{Q_H}) \cdot (\boldsymbol{I} + b\boldsymbol{Q_H^{0.5}})]^{-1} \cdot \boldsymbol{H} \end{aligned} \tag{23}$$

In the code, fractional power of matrix is calculated indirectly using principal matrix logarithm where the matrix function is built on the principal scalar logarithm. The analytical inverse Laplace transform of the equation (20) is:

$$x(t) = e^{a^2 t}\left[b - a\, erf\left(a\sqrt{t}\right)\right] - b e^{b^2 t}\, erfc\left(b\sqrt{t}\right) \tag{24}$$

Error function *erf(x)* and complementary error function *erfc(x)* are two the most widely used functions in science. These functions occur extensively in problems relating to heat conduction, diffusion and probability.

In case of a=0.5 and b=1, for transfer function *X(s)* given by the equation (20), analytical result and numerical results obtained by Haar wavelet method (m=1024), Invlap and NILT algorithms are shown in Fig. 3(a) for interval [0,1) and in Fig. 3(c) for expanded interval $[0,\tau)$. Standard and absolute errors for Haar wavelet method with three different maximum resolution levels (m=64, 256 and 1024) as well as for Invlap and NILT algorithms are presented in Fig. 3(b) for interval [0,1) and in Fig. 3(d) for expanded interval $[0,\tau)$.



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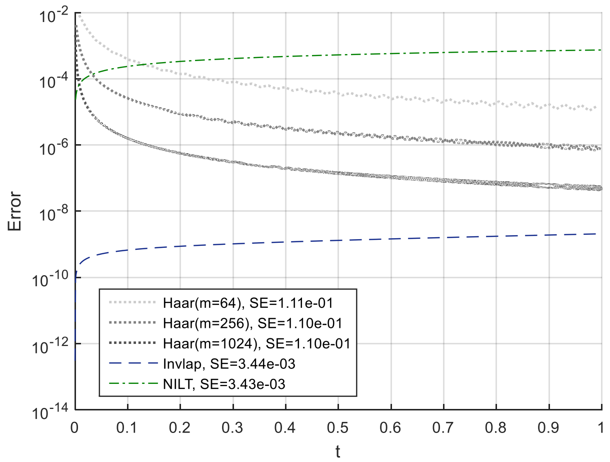

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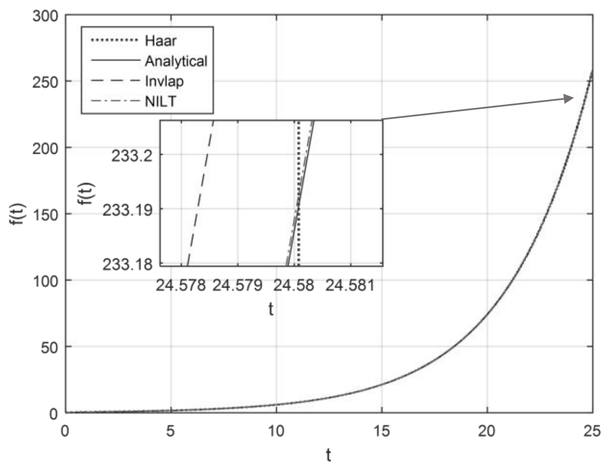

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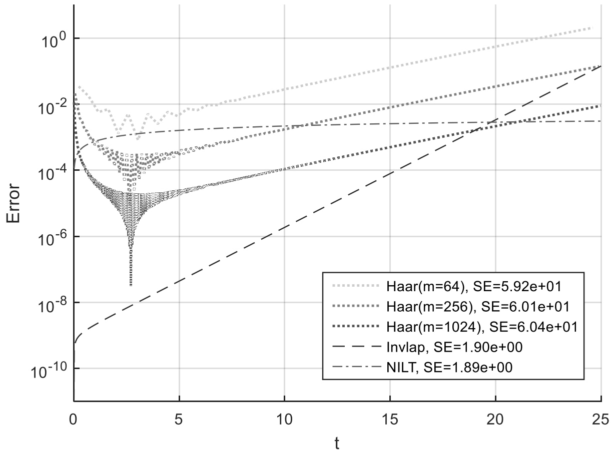

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Fig. 3. The inverse Laplace transform of the transfer function *X(s)* given by the equation (20) obtained analytically and numerically by Haar wavelet method (m=1024), Invlap and NILT algorithms for (a) interval [0,1) and (c) expanded interval $[0,\tau)$. Standard and absolute errors for Haar wavelet method with three different maximum resolution levels and Invlap and NILT algorithms for intervals (b) [0,1) and (d) $[0,\tau)$.

For the transfer function given by the equation (20), in both [0,1) and $[0,\tau)$ interval, Haar wavelet method shows a good agreement with analytical solution as well as Invlap and NILT. When Haar wavelet method is in question, absolute error decreases with time during the [0,1) interval with minimum of the order of 10^{-7} for maximum m value. Standard error is of the order 10^{-1} for all three maximum resolution levels. When expanded interval is in question, after the [0,1) interval, Haar absolute error values fluctuate and have minimum values around t=3 when the increase in error with time starts. Standard error reaches the order of 10^{1} while Invlap and NILT perform slightly better with standard errors of order of 10^{0}. Absolute error of NILT algorithm is almost constant during the whole time and is of order of 10^{-2}. Absolute error of Invlap increases with time. It is in 10^{-10} – 10^{-1} range for $[0, \tau)$ interval.

VI. Accuracy of the Haar Wavelet Method – Heaviside Unit Step Function

Let us consider:

$$X(s) = \frac{1}{s\sqrt{s}}(1 - e^{-Ts}) \tag{25}$$

When we replace *1/s* by Haar wavelet operational matrix $\boldsymbol{Q_H}$:

$$\hat{X}(\boldsymbol{Q_H}) = \boldsymbol{Q_H^{1.5}} \cdot (\boldsymbol{I} - e^{-T\boldsymbol{Q_H^{-1}}}) \tag{26}$$

Then the inversion of Laplace transform can be calculated by:

$$\boldsymbol{x}^{\boldsymbol{T}} = [2m \quad -2m \quad \dots \quad -2m]_{1\times m} \cdot \boldsymbol{H}^T \cdot \boldsymbol{Q}_{\boldsymbol{H}}^{1,5} \cdot (\boldsymbol{I} - e^{-T\boldsymbol{Q}_H^{-1}}) \cdot \boldsymbol{H} \tag{27}$$

The analytical inverse Laplace transform of equation (25) is:

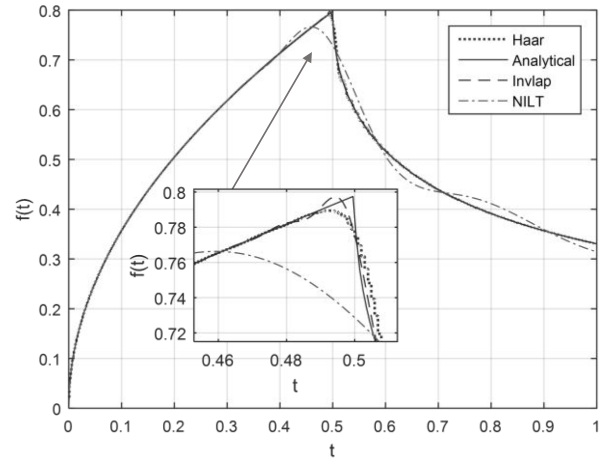
$$x(t) = \frac{2}{\sqrt{\pi}} \cdot \sqrt{x} \cdot H(x) - \sqrt{x-T} \cdot H(x-T) \tag{28}$$

The Heaviside unit step function is used in the signal processing. It represents signals that switch on at specified times and stay switched on indefinitely. It is also used in structural mechanics to describe different structural loads, in engineering where periodic functions are represented, in physics for sudden changes (when breaks are being applied or during collisions), etc.

In case of T=0.5 for transfer function $X(s)$ given by the equation (25), analytical result and numerical results obtained by Haar wavelet method (m=1024), Invlap and NILT algorithms are shown in Fig. 4(a) for interval [0,1) and in Fig. 4(c) for expanded interval $[0,\tau)$. Standard and absolute errors for Haar wavelet method with three different maximum resolution levels (m=64, 256 and 1024) as well as for Invlap and NILT algorithms are presented in Fig. 4(b) for interval [0,1) and in Fig. 4(d) for expanded interval $[0,\tau)$.

When Haar wavelet method is in question, functions with sharp turns are challenging. For the transfer function given by the equation (25), over the [0,1) interval Haar method performs well at the sharp turn, almost as well as Invlap algorithm. Absolute errors at the peak for all algorithms are of the order of 10^{-2}. However, because of the same maximum resolution level value (1024) and a longer period of time, Haar wavelet has poorer performances at the sharp turn than both Invlap and NILT in the $[0,\tau)$ interval. Over the $[0,\tau)$ interval absolute error of Harr wavelet method varied in the 10^{-2} - 10^{-4} range. For NILT algorithm, the absolute error has the highest value. It is of the order of 10^{-2}. Apart from the sharp turn, absolute error of Invlap algorithm is of the order of 10^{-10}. Haar wavelet method standard error is of the order 10^{-1} during the [0,1) interval and 10^{-2} during the $[0,\tau)$ interval for all three maximum resolution levels. Standard errors of Invlap and NILT are of order of 10^{-3} over the whole time span.

From Figs. 1-4 can be seen that numerical inversion Laplace transform using Haar wavelet operational matrix performs very well in case of irrational and transcendental functions. Results obtained by standard and absolute error calculations show that for all examples Haar wavelet standard error values are mainly in the 10^{-1} – 10^{-2} range and that absolute errors depend on the transfer function in question. Accuracy of the numerical solution depend on the value of the maximum resolution level of the operational matrix, especially at sharp turns. Higher values of the parameter m provided better agreement with the analytical solution.



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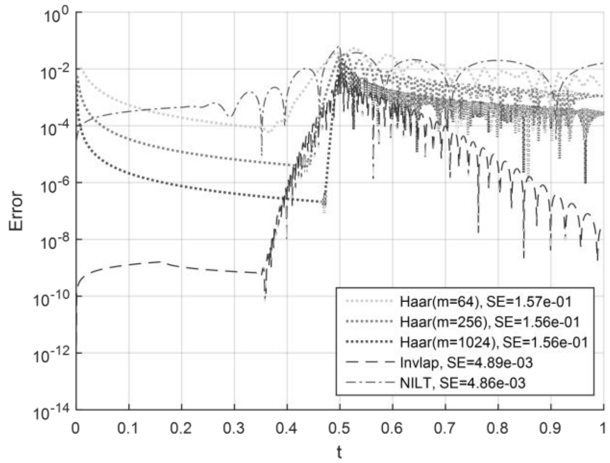

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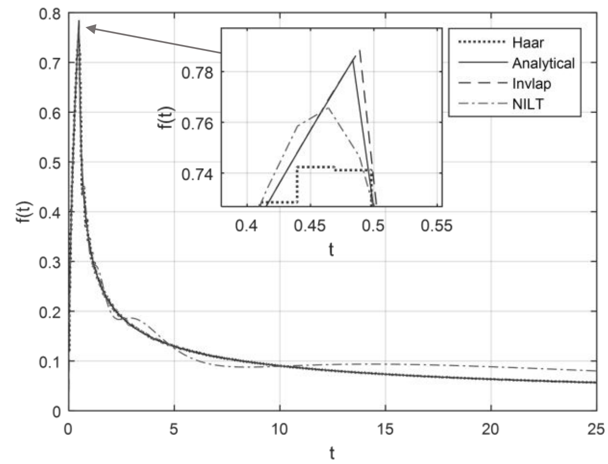

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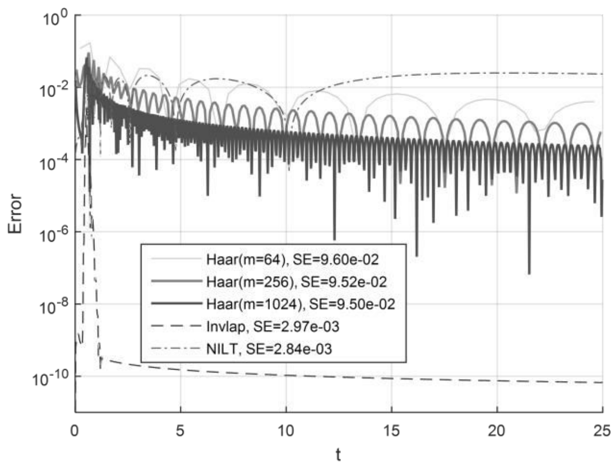

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Fig. 4. The inverse Laplace transform of transfer function obtained analytically and numerically by Haar wavelet method (m=1024), Invlap and NILT algorithms for (a) interval [0,1) and (c) expanded interval $[0,\tau)$. Standard and absolute errors for Haar wavelet method with three different maximum resolution levels and Invlap and NILT algorithms for intervals (b) [0,1) and (d) $[0,\tau)$.

VII. CONCLUSION

In this study accuracy of the Haar wavelet operational matrix application in the inverse Laplace transform numerical calculations for the case of irrational and transcendental transfer functions was investigated. Results for a number of analytically solved inverse Laplace transforms of periodic and non-periodic functions are presented and obtained results are compared with the analytical solutions and results obtained by Invlap and NILT - algorithms that are known to be effective when irrational and transcendental functions are in question. Agreement of the numerical and analytical solutions is quantitatively evaluated using standard and absolute error calculations. For the most of presented examples Haar wavelet method standard error values are in the $10^{-1} – 10^{-2}$ range and absolute errors depend on the transfer function in question. Accuracy of the numerical solution depends on the value of the operational matrix maximum resolution level. Higher values of the operational matrix maximum resolution level improve the agreement between the numerical and analytical solutions especially at sharp turns when longer intervals require higher resolution levels. When compared, all the algorithms used have given acceptable results. Haar wavelet method performed better that NILT for all presented examples. Although Invlap algorithm performed better that Haar wavelet method, results obtained by Haar were in good agreement with analytical solutions for all presented examples. This approach is especially useful when the original cannot be represented by an analytical formula and numerical method must be used. In that case validity of the obtained result can be crosschecked and error can be estimated.

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