Dominating sets on the rhomboidal cactus chains and the icosahedral network

Miroslava Mihajlov Carević[1]
[1] Faculty of Mathematics and Computer Science
Alfa BK University
Beograd, Serbia
miroslava.carevic.mihajlov@alfa.edu.rs

Milena Petrović[2], Nebojša Denić[3]
Faculty of Science and Mathematics
University of Priština
Kosovska Mitrovica, Serbia
milena.petrovic@pr.ac.rs
nebojsa.denic@alfa.edu.rs

Abstract: In this paper, we deal with the dominating set and the domination number of rhomboidal cactus chains. We determine minimum dominating sets and domination numbers of two special types of rhomboidal cactus chains, meta and ortho-chains. Thereafter, we determined the minimum dominating set for the icosahedral hexagonal network.

Keywords*: dominating set; rhomboidal cactus chains; icosahedral hexagonal network.*

I. INTRODUCTION

Like usual in mathematics, we denote the vertex-set and edge-set of graph G by V(G) and E(G) respectively.

A cactus graph is connected graph in which no edge lies in more than one cycle. In the middle of the last century, Husimi [1] began studying these graphs. They have been used in studies of cluster integrals, in the theory of condensation [2], in statistical mechanics [3]. They were later used in the chemistry [4] and in the theory of electrical and communication networks [5].

A subset D of V (G) is called a k-dominating set, if for every vertex y not in D, there exists at least one vertex x in D, such that $d(x, y) \leq k$. With d(x, y) we denote distance between the vertices x and y.

The number of elements of the smallest k-dominating set is called k-domination number and is denoted by γ_k. For $k = 1$, 1-dominating set is called dominating set and 1-domination number is also called domination number and denote by γ.

In the previous period, k-dominance was investigated on triangular [6], rectangular [7] and hexagonal [8] – [12] cactus chains.

In this paper, we analyze the dominating sets of the rhomboidal cactus chains. A rhomboidal cactus G is a graph of a cactus consisting only of cycles of rhomboidal shape that are connected in vertices. A vertex shared by two or more rhomboids is called a cut-vertex. If every rhomboid in G has at most two cut-vertices, and every cut-vertex is shared by exactly two rhomboid, we call G a rhomboidal cactus chain. The number of rhomboids in G is called the length of a chain and denote with h.

If $R^1, R^2, \ldots, R^h$ are consecutive rhomboids in the rhomboidal cactus chain (Figure 1) then we denote the rhomboidal cactus chain by G_h and write

$$G_h = R^1 R^2 \ldots R^h$$

Fig. 1

Let $r_i = \min\{d(x, y) : x \in R^i, y \in R^{i+2}\}$, $i = 1, 2, \ldots, h-2$. We say that r_i is the distance between hexagons R^i and R^{i+2}.

Every rhomboidal cactus chain G_h, $h \geq 2$, has exactly two rhomboids with only one cut-vertex. These are the initial and final rhomboids, R^1 and R^h. All the other rhomboids in a chain have two cut-vertex. They are called internal rhomboids.

An internal rhomboid in G_h is called an ortho-rhomboid if its cut-vertices are adjacent. The distance between its cut-vertices is 1.

An internal rhomboid in G_h is called a meta-rhomboid if the distance between its cut-vertices is 2.

A rhomboidal cactus chains said to be uniform if all its internal rhomboids are of the same type. A chain G_h is called an ortho-chain if all its internal rhomboids are ortho-rhomboids. Analogously, G_h is a meta-chain if all its internal rhomboids are meta-rhomboids.

An ortho-chain of length h we denote by O_h, a meta-chain by M_h. Figure 2 shows the ortho-chain O_8 and Figure 3 the meta-chain M_7.

Fig. 2

Notice that for O_h we have $c_i = 1$, while for M_h we have $c_i = 2$.

Fig. 3

II. DOMINATING SETS ON UNIFORM RHOMBOIDAL CACTUS CHAINS

To determine the dominating set and the domination number of uniform cactus chains O_h and M_h, $h \geq 2$, we will need to label the vertices in the chain.

In the ortho chain the cut-vertices are adjacent and we will denote them by v_i and v_{i+1} while the remaining two vertices will be denoted by x_1^i and x_2^i (Figure 4_A).

In the meta chain, we denote the cut-vertices by v_{i-1} and v_{i+1} and the remaining two vertices by v_i and x_1^i (Figure 4_B).

Fig. 4

Theorem 2.1. For uniform rhomboidal cactus ortho chain

$\gamma(O_h) = h + 1$.

Proof: We are watching the rhomboidal ortho chain

$G_h = R^1 R^2 \ldots R^h$ and the set

$D_{Oh} = \{ x_1^i, i = 1, h\} \cup \{ v_{h+1} \}$. Set D_{Oh} is shown in Fig. 5.

Fig. 5

Let us prove that D_{Oh} is the dominating set of minimal cardinality.

1° If h is even, we will split ortho chain G_h into subchain $R^i R^{i+1}$, i = 1, …, h–1. (Fig. 6)

Fig. 6

Each cycle R^i is of length 4, so it follows that in each cycle R^i there must be one vertex that dominates two adjacent vertices. Vertex x_1^i dominate at v_i and x_2^i. Vertex x_1^{i+1} dominate at v_{i+1} and x_2^{i+1}. In the subchain $R^i R^{i+1}$ there is only vertex v_{i+2}, which was not dominated by vertices x_1^i and x_1^{i+1}. If we add a subchain $R^{i+2}R^{i+3}$, vertex x_1^{i+2} will dominate over v_{i+2} and x_2^{i+2}, vertex x_1^{i+3} will dominate over x_2^{i+3} and v_{i+3} and there will remain vertex v_{i+4} without dominance. We inductively conclude that, along the entire cactus chain, vertices x_1^i and x_1^{i+1} for i = 1, h–1, dominate over other vertices except the vertex v_{h+1}. It follows that the minimal dominating set for O_h is set $D_{Oh} = \{ x_1^i, i = 1, h\}$ $\{ v_{h+1} \}$ and its cardinality is h + 1. So $\gamma(O_h) = h + 1$.

2° If h is odd, we will have one single cycle R^h at the end of the chain (Fig.7).

Fig. 7

As evidenced in the previous section, vertices x_1^{h-2}, x_1^{h-1} i x_1^h dominate over cycles R^{h-2}, R^{h-1}, R^h except over the vertex v_{h+1}. It follows that the minimal dominating set for O_h is set $D_{Oh} = \{ x_1^i, i = 1, h\} \cup \{ v_{h+1} \}$ as in the case of k even number.

Theorem 2.2. For uniform rhomboidal cactus meta chain

$\gamma(O_h) = h + 1$.

Proof: We are watching the rhomboidal meta chain

$G_h = R^1 R^2 \ldots R^h$ and the set

$D_{Mh} = \{ v_{2i-1}, i = 1, h+1\}$. Set D_{Mh} is shown in Fig. 8.

Fig. 8

Similar to the previous one, in every cycle R^i at least 1 vertex is dominant and it dominates the adjacent 2 vertices but not the fourth (not adjacent) vertex. Therefore, this vertex (which is common to two cycles) must be the dominant vertex in the second cycle. We inductively conclude that each R^i in the meta chain $G = R^1 R^2 \ldots R^h$ contains one dominant vertex, but in the last R^h there is no dominance for vertex v_{2h+1}. Therefore, another vertex in R^h must be dominant. It follows that the minimal dominating set for M_h is set $D_{Mh} = \{ v_{2i-1}, i = 1, h+1\}$ and its cardinality is h + 1. So $\gamma(M_h) = h + 1$.

In this case, it is irrelevant that the meta chain contains an even or odd number of cycles (rhomboids), so we do not consider them separately as in the previous case.

III. DOMINATING SET FOR ICOSAHEDRAL-HEXAGONAL NETWORK

It is known that icosahedral structures are present in viruses [13], in molecules of gold [14], copper [15], in methal glasses [16]. The German Weather Service has developed an operational global numerical model of weather forecasting, called GME, based on the icosahedral-hexagonal network [17]. In order to generate a lattice, a regular icosahedron is constructed in the sphere (Fig. 9_A). By successive half edges of the triangles that are sides of the icosahedron, new triangles are formed. The icosahedral network was then developed into a hexagonal network consisting of 10 rhombuses, each consisting of two triangles from the icosahedral network (Fig. 9_B). This network viewed as a graph is the subject of our analysis in this section.

Fig. 9

To determine the dominating set and the domination number of icosahedral hexagonal network we will use the previously discussed considerations and conclusions. We denote vertices in the network with x_1, x_2, ..., x_{20} and rhomboids with $R^1, R^2, \ldots, R^{10}$ (Fig. 10).

Theorem 3.1. For icosahedral hexagonal network GME domination number $\gamma = 6$.

Proof: We are watching the icosahedral hexagonal network and the set

$D = \{ x_0, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}$ shown in the Fig. 10.

Basis on the previously results, we have that vertex x_0 dominate over vertices x_1, x_2, x_3, x_4 and x_5. Vertex x_{16} dominate over vertices x_6 and x_{12}, vertex x_{17} dominate over vertices x_7 and x_{13}, vertex x_{18} dominate over vertices x_8 and

x_{14} , vertex x_{19} dominate over vertices x_9 and x_{15} , vertex x_{20} dominate over vertices x_{10} and x_{11}. It follows that set $D = \{ x_0, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}$ is the minimal dominating set of icosahedral hexagonal network GME. Therefore, domination number $\gamma = 6$.

REFFERENCES

[1] K. Husimi, *Note on Mayer's theory of cluster integrals*, J. Chem. Phys. 18 (1950), 682–684.

[2] R. J. Riddell, *Contributions to the theory of condensation*, Ph. D. thesis, Univ. of Michigan, Ann Arbor, 1951.

[3] G. E. Uhlenbeck and G. W. Ford, *Lectures in Statistical Mechanics*, AMS, Providence RI, 1956.

[4] B. Zmazek and J. Žerovnik, *Computing the weighted Wiener and Szeged number on weighted cactus graphs in linear time,* Croat. Chem. Acta 76 (2003), 137–143.

[5] B. Zmazek and J. Žerovnik, *Estimating the traffic on weighted cactus networks in linear time*, Ninth International Conference on Information Visualisation, (IV'05) (2005), 536–541.

[6] E. J. Farrell, *Matchings in triangular cacti*, Math. Sci. (Calcutta), 11 (2000), 85–98.

[7] E. J. Farrell, *Matchings in rectangular cacti,* Math. Sci. (Calcutta), 9 (1998), 163–183.

[8] E. J. Farrell, *Matchings in hexagonal cacti,* Internat. J. Math. & Math. Sci. 10, (1987), 321–338.

[9] S. Majstorovic, T. Doslic, and A. Klobucar, *k-Domination on hexagonal cactus chains*, Kragujevac journal of mathematics, (2012), 36(2):335-47.

[10] T. Došlić and F. Måløy, *Chain hexagonal cacti: Matchings and independent sets*, Discrete mathematics, (2010), 310(12):1676-90.

[11] Z. Yarahmadi, T. Došlić and S. Moradi, *Chain hexagonal cacti: extremal with respect to the eccentric connectivity index*, Iranian Journal of Mathematical Chemistry, (2013), 4(1):123-36.

[12] A. Klobučar and A. Klobučar, *Total and Double Total Domination Number on Hexagonal Grid*, Mathematics, 2019, *7*(11):1110.

[13] R. Zandi, D. Reguera, R.F. Bruinsma, W.M. Gelbart and J. Rudnick, *Origin of icosahdral symmetry in viruses,* Proceedings of the National Academy of Science, (2004). 101(44): 15556-15560.2004.

[14] Y. Wang, S. Teitel and C. Dellago, *Melting of icosahedral gold nanoclusters from molecular dynamics simulations,* The Journal of chemical physics, (2005) 122(21), 214722.

[15] P. Ganesh and M. Widom, *Signature of nearly icosahedral structures in liquid and supercooled liquid copper,* Physical Review B. (2006). 74(13), 134205.

[16] R. Soklaski, Z. Nussinov, Z. Markow, F. Kelton and L. Yang, *Connectivity of icosahedral network and a dramatically growing static length scale in Cu-Zr binary metallic glasse,* Physical Review (2013), 87(18), 184203.

[17] D. Majewski, D. Liermann, P. Prohl, B. Ritter, M. Buchhold, T. Hanisch and J. Baumgardner, *The operational global icosahedral–hexagonal gridpoint model GME: description and high-resolution tests,* Monthly Weather Review, (2002), 130(2): 319-338.

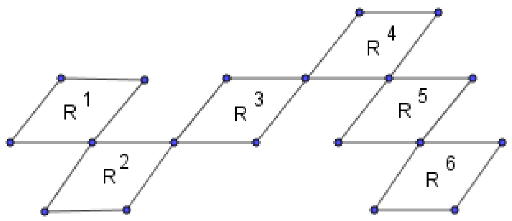


Fig. 1. Rhomboidal cactus chain of length 6

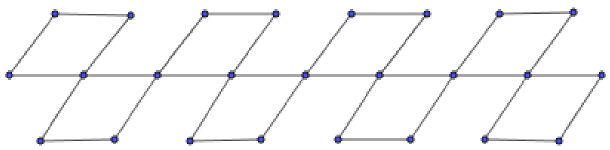


Fig. 2. Ortho-chain of length 8

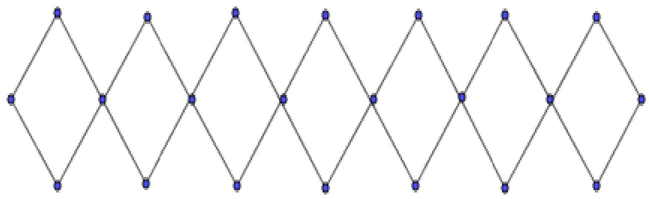
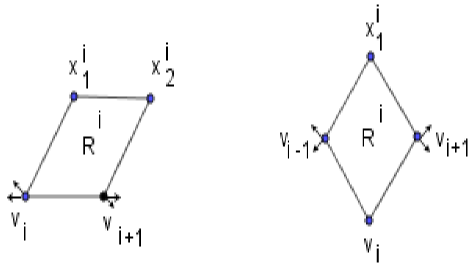


Fig. 3. Meta-chain of length 7



4_A 4_B

Fig. 4. Labeling of vertices in uniform ortho and meta chains

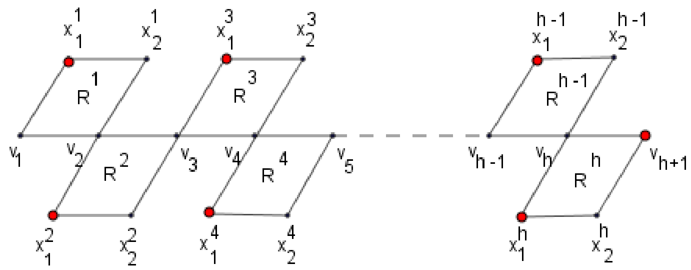


Fig. 5. Minimal dominating set of Oh

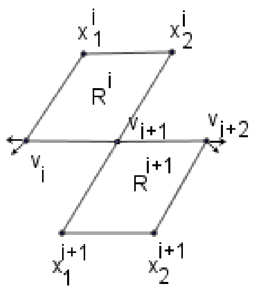


Fig. 6. A subchain of the ortho chain

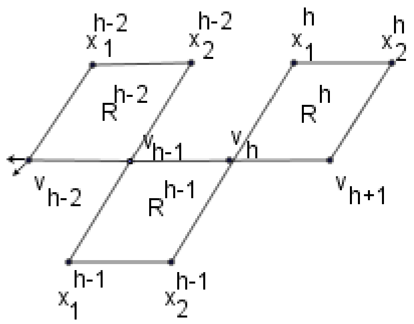


Fig. 7. The single cycle R^h at the end of the chain

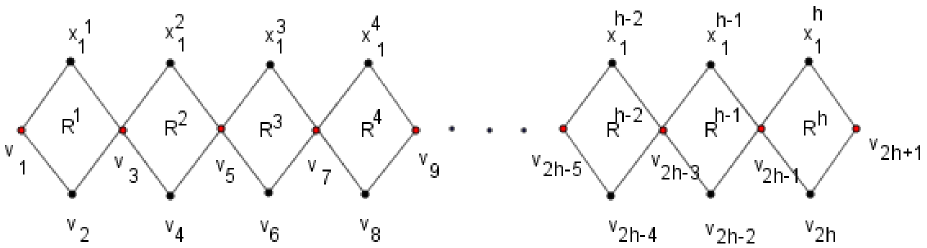
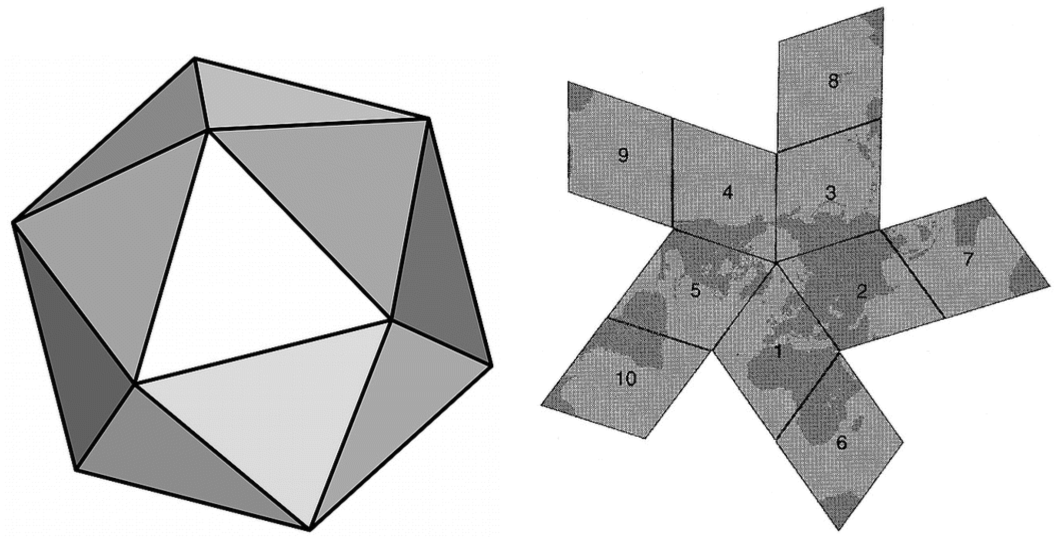


Fig. 8. Minimal dominating set of Mh.



9_A 9_B

Fig. 9. Icosahedron and icosahedral hexagonal GME network

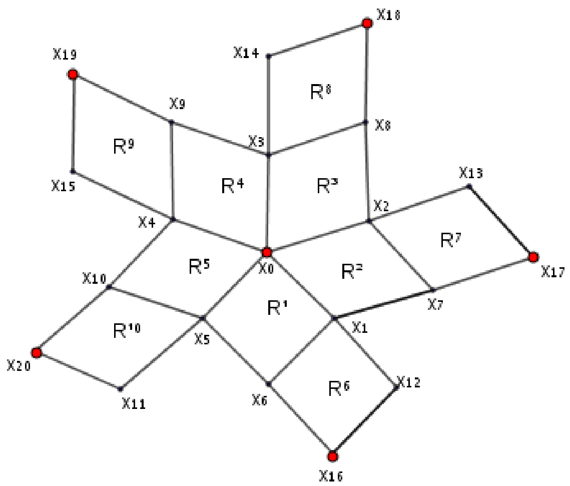


Figure 10. Minimal dominating set of icosahedral hexagonal network