18th International Symposium INFOTEH-JAHORINA, 20-22 March 2019

Ratio of the Product of Weibull and Gamma Random Variables for Modelling Fading Environment

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Abstract—**In the presented paper, effects on the envelope of signal in fading environment are modeled as exact probability density function (PDF) and cumulative distribution function (CDF) of the ratio of product of two independent stable Weibull random variables and Gamma random variables. Those functions are derived in terms of the Meier-G function. Especially important value for the systems analysis modeled by those ratios of products is CDF. CDF are reliable model for calculation of system failure. Simple simulation routine in Mathematica software has been developed for the evaluation delivered PDF and CDF in term of Meier-G function. Simulation complete numerical experiments to show the accuracy and correctness of the expressions hereby deduced.**

Keywords: Weibull distribution, Gamma Distribution, Fading

I. Introduction

Products and ratios of two random variable, where those variables are mutually independent and belong to same or different types of mathematical functions, have been studied in engineering intensively in recent years. Those products and ratios of random distributed variables are applicable in the different fields of modeling real systems.

Calculation of joint probability density function PDF of those ratios and products is first step in calculation and derivation of closed forms of cumulative distribution function CDF. CDF can provide important information on the system behavior. One of the most information acquired from CDF is outage probability or probability of failing of system [1].

Ratios and products of random variables are applied in the different fields of engineering. Telecommunication engineering those ratios usually apply in the analysis of behavior of communication systems in different environmental modes. Closed solutions of ratio and products are used to analyze outage probability, delay-limited and ergodic capacities and different physical channel modes [2].

Products of two Parreto distributions can be applied to model hydrological problems, while product of two extreme value distributions can be used to calculate drought analysis [3-4].

Economy, and finance use heavy tailed distributions as established as standard models. Random variable types used in economics are: Parreto type variables, extreme value distributions and alpha stable distributions [5-7].

One of the standard distributions used for modeling of the real time communication system is Weibull distribution. System explained with Weibull random variable have self-disruptive attributes. During propagation of signal envelope through channel Weibull distributed variable model fast fading on signal.

While one sole function explains self-disruptive effects of fading on signal envelope, ratios and products of random variables are used to describe any additional disruptive effect on modeled system. Ratio of product explains disruptive effects of co channel interference CCI, byproduct of reuse of frequencies. Product of two random variables describes effects of slow fading effects on envelope of the signal.

We can use product of same functions for modeling, like product of two Weibull random variables. Those models are up to some level applicable to realistic scenarios. More realistic results are obtained when we use Gamma distributed function for modeling slow fading effects [8].

In this paper we will deliver closed form equations for the PDF and CDF of the ratio of product of Weibull and Gamma distributed functions in terms of Meier-G functions. Modern mathematical software can solve numerical those ratios, without closed form solutions. Those numerical calculation need reasonable calculating time. Calculation of closed form is faster and more reliable.

Simulation routine in the Mathematica software package has been programmed in order to provide numerical experiment.

Numerical experiments are carried out to show the accuracy and correctness of the expressions hereby deduced. Derived results and closed form are not published before to the best knowledge of author.

II. Statistics of Ratio of Products of Weibull and Gamma Random Variables

Weibull random variable can describe statistical behavior of arbitrary random process with self-disruptive properties with following equation.

$$p_\gamma(\gamma) = \frac{\beta}{\Omega_\gamma}\gamma^{\beta-1}e^{-\frac{\gamma^\beta}{\Omega_\gamma}} \tag{1}$$

In telecommunication system in fading environment this equation presents envelope of the signal in fast fading environment. Fast fading occur as result of reflection and refraction of the signal in environment. Value β represents fading coefficient of the system. Higher value of β present lower fading effects on the envelope of the signal.

Second disruptive effect on the signal envelope based on slow fading effects is presented with Gamma function. Slow fading effects are result of obstacles between transmitter and receiver in telecommunication system.

$$p_y\left(y\right) = \frac{y^{c-1}\exp\left(-y/y_0\right)}{\Gamma\left(c\right)y_0{}^c} \tag{2}$$

In this model severity of gamma shadowing is measured in terms of c. The lower value of c means the higher shadowing effects. Those effects we will apply on envelope of useful signal on upper part of ratio expression and on CCI in lower part of ratio expression.

Statistical analysis of ratio of products of Weibull and Gamma random variables will be delivered in the form of ʋ=xy/zw.

.For derivation of the corresponding PDF of ʋ, we have to obtain the PDF of ratio of two alpha mu random variables, λ=x/z, using the following equation

$$p_\lambda(\lambda) = \int_0^\infty |J| p_x(\lambda z)p_z(z)dz \tag{3}$$

Where |J| is the Jacobian transformation given by |J|=|dx/dλ|=z. Applying [[9], (3.478-1)],[10] and simple mathematical transformation joint PDF can be delivered in terms of Meier-G function

$$p_\lambda\left(\lambda\right) = \beta\left(\frac{\Omega_y}{\Omega_x}\right)\lambda^{\beta\mu_x-1}\left(\frac{\Omega_y\lambda^\beta}{\Omega_x}+1\right)^{-2} \tag{4}$$

Using similar procedure, the PDF of ratio of two gamma random variables λ=y/w can be derived as

$$p_t(t) = \int_0^\infty |J|\, p_y(\frac{t}{w})p_w(w)dw \tag{5}$$

$$p_t(t) = \frac{t^{c-1}\Omega_w^c}{(\Gamma(\mathrm{c}))^2\Omega_y^c}G_{1,1}^{1,1}\left(\frac{\Omega_w t}{\Omega_y}\lambda^2\middle|\begin{matrix}1-2c\\0\end{matrix}\right) \tag{6}$$

In order to obtain joint PDF for random variable ʋ we use equation for joint probability of the product of two random variables.

$$p_\nu\left(\gamma\right) = \int_0^\infty \frac{1}{t}p_\lambda\left(\frac{\nu}{y}\right)p_t\left(t\right)dt \tag{7}$$

Using [[9], (3.478-1)],[10] we deliver PDF of the ratio of the product of two random variables in terms of Meier-G function.

$$\begin{aligned}p_\nu(\nu) = {} & \left(\frac{\Omega_z}{\Omega_x}\right)\left(\frac{\Omega_w}{\Omega_y}\right)^\beta\frac{\nu^{\beta\mu_x-1}\beta^{2c}}{\left(2\pi\right)^{\alpha-1}(\Gamma(c))^2}\times\\ & \times G_{1+\beta,1+\beta}^{1+\beta,1+\beta}\left(\frac{1}{\nu}\frac{\Omega_x}{\Omega_z}\left(\frac{\Omega_y}{\Omega_w}\right)^\beta\middle|\begin{matrix}1,\frac{1}{\beta}(\beta\mu_x-c+1),\ldots,\frac{1}{\beta}(\beta(\mu_x+1)-c)\\2,\frac{1}{\beta}(\beta\mu_x+c),\ldots,\frac{1}{\beta}(\beta(\mu_x+1)+c-1)\end{matrix}\right)\end{aligned} \tag{8}$$

CDF of ratio of product random variable ʋ can be calculated by definition as

$$F_V\left(\nu\right) = \int_0^\nu p_V\left(s\right)ds \tag{9}$$

After applying the described procedure, with the aid of [9, eqs. (3.461), (6.631) and 7.813 (1)]], [10, eq. (26)], CDF of λ is expressed in terms of Meijer G functions.

$$\begin{aligned}F_\nu(\nu) = {} & \frac{\Omega_z}{\Omega_x}\left(\frac{\Omega_w}{\Omega_y}\right)^\beta\frac{\nu^\beta\beta^{2c-1}}{\left(2\pi\right)^{\beta-1}(\Gamma(c))^2}\times\\ & \times G_{2+\beta,2+\beta}^{1+\beta,2+\beta}\left(\frac{\Omega_z}{\Omega_x}\left(\frac{\Omega_w}{\Omega_y}\right)^\beta\nu^\beta\middle|\begin{matrix}-1,1-\frac{1}{\beta}(\beta+c),\ldots,1-\frac{1}{\beta}(2\beta+c-1),0\\0,\frac{1}{\beta}(\beta_x-c+1),\ldots,1-\frac{1}{\beta}(2\beta-c-1),-1\end{matrix}\right)\end{aligned} \tag{10}$$

III. MATHEMATICAL SIMULATION OF DELIVERED CLOSED FORM SOLUTIONS

Simulation of complex functions can be cumbersome. Latest distribution of the mathematical software like Mathematica have developed routines for calculation of confluent hypergeometric functions.

One of the prominent functions of this type are Meier-G functions. Implementation of those routines provided us tool for simulation and calculation of the values of PDF and CDF.

Values of the parameters which describe fading effects in environment can be easily adjusted. We also can change values of function values which represent power of useful or disruptive signal.

Following code is simple code developed in the package Wolfram Mathematica for calculation of the ratio of products of Weibull and Gamma random variables.

```
veraspodelaA[ mx_,my_,mz_,mw_,c_,v_,Ωx_,Ωy_,Ωz_,Ωw_]:=

((mx  Ωz)/(mz  Ωx))mx  (Ωw/Ωy)2mx  (v2  mx  22  c)/(2  Ω
Gamma[mx] Gamma[mz] (Gamma[c])2)

MeijerG[{{1-mx-mz,1-1/2 (2 mx+c),1-1/2 (2 (mx+1)+c-1),1-
2mx},{}},{{0,1-1/2 (2 mx-c+1),1-1/2 (2(mx+1)-c)},{-2mx} },(v
mx Ωz)/(mz Ωx) (Ωw/Ωy)2];

veraspodelaB[ mx_,my_,mz_,mw_,c_,v_,Ωx_,Ωy_,Ωz_,Ωw_]:=
```

```
    ((mx  Ωz)/(mz  Ωx))mx  (Ωw/Ωy)3mx  (v3  mx  32  c)/((2  Ω)2
Gamma[mx] Gamma[mz] (Gamma[c])2)

    MeijerG[{{1-mx-mz,1-1/3  (3  mx+c),1-1/3  (3  mx+c+1),1-1/3
(3  (mx+1)+c-1),1-3mx},{}},{{0,1-1/3  (3  mx-c+1),1-1/3  (3  mx-
c+2),1-1/3   (3(mx+1)-c)},{-3mx}   },(v   mx   Ωz)/(mz   Ωx)
(Ωw/Ωy)3];

    xval=Table[i,{i,-15,15}];

    v10A2=Table[0,{i,1,Length[vAr]}];

    v7A2=Table[0,{i,1,Length[vAr]}];

    v5A2=Table[0,{i,1,Length[vAr]}];

    v3A2=Table[0,{i,1,Length[vAr]}];

    v10A3=Table[0,{i,1,Length[vAr]}];

    v7A3=Table[0,{i,1,Length[vAr]}];

    v5A3=Table[0,{i,1,Length[vAr]}];

    v3A3=Table[0,{i,1,Length[vAr]}];

    For[i = 1, i <= Length[vAr], i++,

    v10A2[[i]] = N[veraspodelaA[1, 1, 1, 1, 3, vAr[[i]], 10,
1, 1, 1]];

    v7A2[[i]] = N[veraspodelaA[1, 1, 1, 1, 3, vAr[[i]], 7, 1,
1, 1]];

    v5A2[[i]] = N[veraspodelaA[1, 1, 1, 1, 3, vAr[[i]], 5, 1,
1, 1]];

    v3A2[[i]] = N[veraspodelaA[1, 1, 1, 1, 3, vAr[[i]], 3, 1,
1, 1]];

    v10A3[[i]] = N[veraspodelaB[1, 1, 1, 1, 3, vAr[[i]], 10,
1, 1, 1]];

    v7A3[[i]] = N[veraspodelaB[1, 1, 1, 1, 3, vAr[[i]], 7, 1,
1, 1]];

    v5A3[[i]] = N[veraspodelaB[1, 1, 1, 1, 3, vAr[[i]], 5, 1,
1, 1]];

    v3A3[[i]] = N[veraspodelaB[1, 1, 1, 1, 3, vAr[[i]], 3, 1,
1, 1]];

    ];

    res=Transpose[{xval,v10A2,v7A2,v5A2,v3A2,v10A3,v7A3,v5A3,
v3A3}];or

    Export["vejblData.xls",res]
```

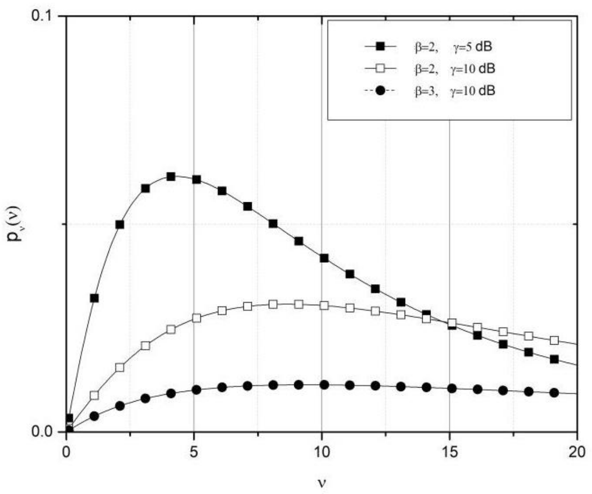

Figure 1. PDF of the ratio of the product of Weibull and Gamma function

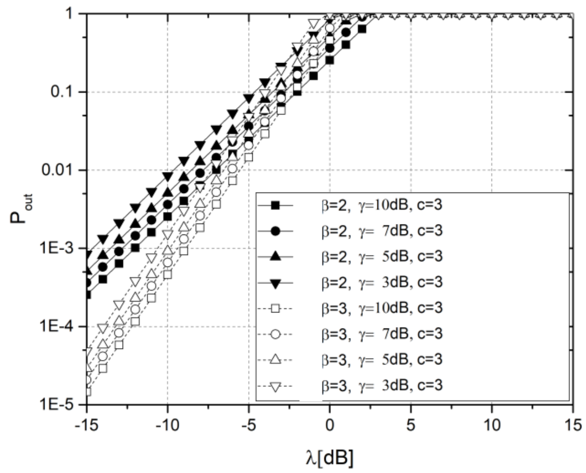

Figure 2. CDF of the ratio of the product of Weibull and Gamma function

Using this simulation we delivered PDF and CDF model for cases when we increase fading severity and decrease value of useful signal. Analysis of the simulation delivered for CDF showed that outage probability is increased in cases of harsher fading environment. Also this outage probability is increased when power of useful signal is decreased while disruptive signal envelopes are increased. Results of numerical simulation, is in line with theoretical fading models, and are given on Fig.1. and Fig.2.

IV. CONCLUSION

Deriving of the closed forms of the statistical ratios and products is important tool in modeling any complex system which follows this statistical nature.

Derived closed forms can be used in and statistical environment were disruptive effects are described with Weibull or Gamma random variable. Our case analyzed models of fading were self-disruptive effects were modeled with Weibull random variable, while additional disruptive effects on the random process were modeled with Gamma random variable. Complex realistic model also included disruptive effects of the additional process as lower part of the ratio expression.

Analysis showed that exact closed form expressions for products and ratios are easy for calculation in modern mathematical software. Implementation of routines for calculation of complex functions provided powerful tool for modeling of systems. Numerical calculation of the presented models are processor burdening and we need to dedicate significant calculating power to receive approximate solution.

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