Level crossing rate of the product of two random variables

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Abstract—**In this paper level crossing rate (LCR) of the product of two random variables is considered. Joint probability density function of product of two random variables and its first time derivate are also calculated. Random variables follow Nakagami-m, α-μ, Rayleigh and Weibull probability distributions. Expressions for LCR of product of two random variables can be used for evaluation of the average fade duration of wireless communication system, operating over multipath fading channels. Numerical results are graphically presented to show the influence of fading parameters on system performances**

Keywords 1; Nakagami-m fading 2; joint probability density function, 3; level crossing rate

I. INTRODUCTION

In this paper the product of two random variables is considered. Statistics of the product of two random variables is important in performance analysis of wireless communication systems operating over fading channels [1-2]. Random variables included in this paper follow Nakagammi–m distribution. Nakagammi–m distribution can be used to describe signal envelope variation subjected to multipath fading. The distribution has two parameters. The parameter m is severity of fading and this parameter is related to the number of clusters of propagation wave. The parameter Ω is average squared value of signal envelope. Nakagami–m distribution is general distribution. For m=1, Nakagammi–m distribution reduces to Rayleigh distribution. For parameter m=0.5, Nakagammi–m distribution reduces to one- side Gaussian distribution. In this work, the probability density function of the product of two Nakagammi–m random variables is calculated. This result can be used for determination of the bit error probability, outage probability and channel capacity in wireless communication systems operating over Nakagammi–m fading channels. One variable can be used to represent small scale variation of signal envelope and another variable can be used to describe large scale variation of signal envelope [3]. Both variables can be used to describe small scale variation of signal envelope but also, both variables can be used to describe large scale of signal envelope. Statistics of the product of two random variables can be used to describe the variation of signal envelope in composite fading environment [7], [9]. On the similar manner the analysis can be done, when the random variables follow α-μ distribution, Rayleigh distribution and Weibull distribution. In this paper, the joint probability density function (JPDF) of the product of two Nakagammi–m random variables and its derivative is calculated. The expression for this joint distribution can be used for determination of the level crossing rate of wireless communication system operating over Nakagammi–m · Nakagammi–m fading environment [4-5]. The expression for the level crossing rate of the product of two random variables can be used for evaluation of average fade duration, of wireless communication systems operating over Nakagammi–m · Nakagammi–m fading channels [6]. This work is organized as follow. In the section II, the probability density function of two random variables is derived. In section III, the joint probability density function of the product of two random variables and its first derivative is calculated. In section IV, average level crossing rate (LCR) of the product of two random variables is investigated.

II. THE PRODUCT OF TWO NAKAGAMMI-M RANDOM VARIABLES

Random variables x and y follow Nakagami-m distribution

$$P_x(x) = \left(\frac{m_1}{\Omega_1}\right)^{m_1} \frac{2}{\Gamma(m_1)} x^{m_1-1} e^{-\frac{m_1}{\Omega_1}x^2} \quad (1)$$

$$P_x(y) = \left(\frac{m_2}{\Omega_2}\right)^{m_2} \frac{2}{\Gamma(m_2)} y^{m_2-1} e^{-\frac{m_2}{\Omega_2}y^2} \quad (2)$$

where m_1 and m_2 are parameters describing the severity of fadings and Ω_1 and Ω_2 are averaged square values of random envelopes x and y. Random variable z is product of two random variables x and y.

$$z = xy,\ x = \frac{z}{y} \quad (3)$$

The random variables can represent random signal envelopes of wireless communication system subjected simultaneously to independent Nakagami-m multipath fadings. The conditional probability density function of z is

$$p_z\left(\frac{z}{y}\right) = \left|\frac{dx}{dz}\right| p_x\left(\frac{z}{y}\right) \quad (4)$$

where $\frac{dx}{dz} = \frac{1}{y}$

By averaging the expression (4) with respect to y the probability density function of random variables z can be written in the form.

$$p_z(z) = \int_0^\infty \frac{1}{y} p_x\left(\frac{z}{y}\right) p_y(y) dy \tag{5}$$

By substituting the expressions (1-2) in (5), the previously expression becomes

$$p_z(z) = \frac{4}{\Gamma(m_1)\Gamma(m_2)} \left(\frac{m_1}{\Omega_1}\right)^{m_1} \left(\frac{m_2}{\Omega_2}\right)^{m_2}$$

$$* \int_0^\infty \frac{1}{y} \left(\frac{z}{y}\right)^{m_1 - 1} e^{-\frac{m_1 z^2}{\Omega_1 y^2}} e^{-\frac{m_2}{\Omega_2} y^2}$$

$$= \frac{4z^{2m_1-1}}{\Gamma(m_1)\Gamma(m_2)} \left(\frac{m_1}{\Omega_1}\right)^{m_1} \left(\frac{m_2}{\Omega_2}\right)^{m_2}$$

$$* \int_0^\infty y^{2m_2 - 2m_1 - 1} e^{-\frac{m_1 z^2}{\Omega_1 y^2} - \frac{m_2}{\Omega_2} y^2} dy \tag{6}$$

Using the integral

$$\int_0^\infty x^{v-1} e^{-\beta x^p - \gamma x^{-p}} dx = \frac{2}{p} \left(\frac{\gamma}{\beta}\right)^{\frac{v}{2p}} K_{\frac{v}{p}}(2\sqrt{\beta\gamma})$$

$$= \frac{2}{p} \left(\frac{\gamma}{\beta}\right)^{\frac{v}{2p}} K_{\frac{v}{p}}(2\sqrt{\beta\gamma}) \tag{7}$$

The expression for probability density function (6) becomes

$$p_z(z) = \frac{4}{\Gamma(m_1)\Gamma(m_2)} \left(\frac{m_1}{\Omega_1}\right)^{m_1} \left(\frac{m_2}{\Omega_2}\right)^{m_2}$$

$$* \left(\frac{m_1 z^2 \Omega_2}{m_2 \Omega_1}\right)^{\frac{m_2 - m_1}{2}} K_{m_2 - m_1}\left(2\sqrt{\left(\frac{m_1 z^2 m_2}{\Omega_1 \Omega_2}\right)}\right) \tag{8}$$

III. JPDF OF THE PRODUCT OF TWO RANDOM VARIABLES AND ITS FIRST DERIVATIVE

The expression for probability density function of products of two Nakagami-m random variables can be used for calculation the bit error probability and outage probability of wireless communication system operating over Nakagami-m multipath channels. The first derivative of the product of two Nakagami-m random variables is

$$\dot{z} = \dot{x}y + x\dot{y} \tag{9}$$

Random variables $\dot{x}$ and $\dot{y}$ follow Gaussian distribution. Therefore $\dot{z}$ has also conditional Gaussian distribution with zero mean. The variation of $\dot{z}$ is

$$\delta_{\dot{z}}{}^2 = y^2 \delta_{\dot{x}}{}^2 + x^2 \delta_{\dot{y}}{}^2 \tag{10}$$

The variations of $\dot{x}$ and $\dot{y}$ are

$$\delta_{\dot{x}}{}^2 = \pi^2 f_m{}^2 \Omega_1 = k\Omega_1 \tag{11}$$

$$\delta_{\dot{y}}{}^2 = \pi^2 f_m{}^2 \Omega_2 = k\Omega_2 \tag{12}$$

where f_m is maximal Doppler frequency and k = $\pi^2 f_m{}^2$

The variance of the first derivative of z is

$$\delta_{\dot{z}}{}^2 = y^2 k\Omega_1 + \frac{z^2}{y^2} k\Omega_2 = \frac{k}{y^2}(y^4\Omega_1 + \Omega_2 z^2) \tag{13}$$

The conditional probability density function of $\dot{z}$ is

$$p_{\dot{z}}\left(\frac{\dot{z}}{yz}\right) = \frac{y}{\sqrt{2\pi k(y^4\Omega_1 + \Omega_2 z^2)}} e^{-\frac{\dot{z}^2}{2k(y^4\Omega_1 + \Omega_2 z^2)} y^2} \tag{14}$$

The joint probability density function of the product of the two random variables z, the first derivative of the product of two random variables $\dot{z}$ and the random variable y is

$$p_{z\dot{z}y}(z\dot{z}y) = p_{\dot{z}}\left(\frac{\dot{z}}{yz}\right) p_{zy}(zy)$$

$$= p_{\dot{z}}\left(\frac{\dot{z}}{yz}\right) p_y(y) p_z\left(\frac{z}{y}\right) \tag{15}$$

The conditional probability density function of z is

$$p_{\dot{z}}\left(\frac{\dot{z}}{yz}\right) = \left|\frac{dx}{dy}\right| p_x\left(\frac{z}{y}\right) \tag{16}$$

where $\frac{dx}{dy} = \frac{1}{y}$

Joint probability density function of $z\dot{z}y$ is

$$p_{z\dot{z}y}(z\dot{z}y) = p_{\dot{z}}\left(\frac{\dot{z}}{yz}\right) \frac{1}{y} p_x\left(\frac{z}{y}\right) p_y(y)$$

$$= \frac{1}{y\sqrt{2\pi}\,\delta_{\dot{z}}} e^{-\frac{\dot{z}^2}{2\,\delta_{\dot{z}}{}^2}} p_x\left(\frac{z}{y}\right) p_y(y) \tag{17}$$

By integrating previous expression with respect to y the joint probability density function of z and $\dot{z}$ becomes

$$p_{z\dot{z}}(z\dot{z}) = \int_0^\infty \frac{1}{y} \frac{1}{\sqrt{2\pi}\,\delta_{\dot{z}}} e^{-\frac{\dot{z}^2}{2\,\delta_{\dot{z}}{}^2}} p_x\left(\frac{z}{y}\right) p_y(y) dy \tag{18}$$

IV. LEVEL CROSSING RATE

The level crossing rate of z is average value of $\dot{z}$

$$N_z = \int_0^\infty \dot{z} p_{z\dot{z}}(z\dot{z})\, d\dot{z}$$

$$= \int_0^\infty \frac{1}{y} \frac{\delta_{\dot{z}}}{\sqrt{2\pi}} p_x\left(\frac{z}{y}\right) p_y(y) dy$$

$$= \frac{z^{2m_1 - 1}\sqrt{k}}{\sqrt{2\pi}\Gamma(m_1)\Gamma(m_2)} \left(\frac{m_1}{\Omega_1}\right)^{m_1} \left(\frac{m_2}{\Omega_2}\right)^{m_2}$$

$$* \int_0^\infty y^{2m_2 - 2m_1 - 2} \sqrt{(y^4\Omega_1 + \Omega_2 z^\alpha)} e^{-\frac{m_1 z^2}{\Omega_1 y^2} - \frac{m_2}{\Omega_2} y^2} dy \tag{19}$$

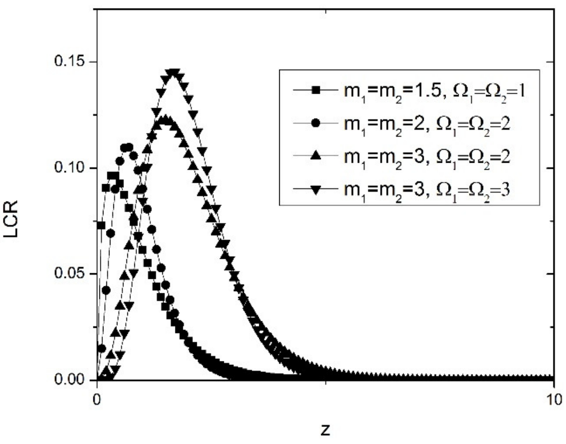

Figure 1. LCR.

The curves show normalized level crossing rate in terms of product of two Nakagammi – m random variables illustrated in Figure 1. The Figure shows results for different values of parameters m and Ω. When the fading severity increases, maximum of the curve increases and as envelope power increases, also, maximum of the curve increases. Furthermore, when the fading severity increases the curves spread.

Obtained expression for the average level crossing rate of the product of two Nakagammi –m random variables can be used for determination of level crossing rate of output signal of wireless communication system operating over Nakagammi – m * Nakagammi – m multipath fading environment. Obtained expression also can be used for calculation of fade duration of wireless system which is simultaneously affected by two Nakagammi – m multipath fadings. The average fade duration can be calculated as ratio of outage probability and average level crossing rate. The outage probability is probability that output of the signal falls below of predetermined outage threshold.

On the similar manner, expressions for the level crossing rate of the product of two α-μ random variables, two Rayleigh random variables and two Weibull random variables are obtained.

Level crossing rate of the product of two α-μ random variables is

$$N_z = \frac{z^{\alpha m_1 - \frac{\alpha}{2} * k}}{\sqrt{2\pi}\Gamma(m_1)\Gamma(m_2)} \left(\frac{m_1}{\Omega_1}\right)^{m_1} \left(\frac{m_2}{\Omega_2}\right)^{m_2}$$

$$* \int_0^\infty y^{2m_2-2m_1-2} \sqrt{(y^4\Omega_1 + \Omega_2 z^\alpha)}\, e^{-\frac{m_1 z^2}{\Omega_1 y^2} - \frac{m_2}{\Omega_2} y^2} dy \quad (20)$$

Level crossing rate of the product of two Rayleigh random variables is

$$N_z = \frac{4}{\sqrt{2\pi}\Omega_1\Omega_2} k_0 \left(\frac{2z}{\sqrt{\Omega_1\Omega_2}}\right) \frac{1}{y} \sqrt{(y^4\Omega_1 + \Omega_2 z^2)} \quad (21)$$

Level crossing rate of the product of two Weibull random variables is

$$N_z = \frac{2\sqrt{k}}{\sqrt{2\pi}\,\Omega_1\Omega_2} z^{\alpha-1}$$

$$* \int_0^\infty \frac{1}{y} \sqrt{(y^{2\alpha} z^{2-\alpha}\Omega_1 + \Omega_2 z^2)}\, e^{-\frac{y}{\Omega_2}^{\alpha} - \frac{z^\alpha}{\Omega_1 y^\alpha}} \frac{1}{\frac{\alpha}{y^2}} dy \quad (22)$$

V. CONCLUSION

In this work, the product of two random variables is analyzed. For this product, the probability density function is calculated in closed form for the case when random variables follow Nakagammi – m distribution. This result can be used for evaluation of the bit error probability, outage probability, channel capacity and amount of fading of wireless communication systems operating over Nakagammi – m* Nakagammi – m multipath fading. Then, in this paper, the joint probability density function of the product of two random variables and the first derivative of the first random variables is calculated. The expression has one integral. The joint probability density function of the product of two random variables can be used for calculation of level crossing rate of output signal of wirelees communication system operating over Nakagami – m * Nakagammi – m multipath fading environment. Furthermore, the level crossing rate of the product of two random variables is determined. This expression has also, one integral and can be used for calculation of average fade duration of wireless communication systems subjected simultaneously to two multipath fadings. The level crossing rate of the product of two random variables is calculated and the cases when random variables follow α-μ, Rayleigh and Weibull distribution. The results attained in this paper can be applied in performance analysis twohop relayed wireless communication systems when the signal level is sufficiently greater than the noise level, so the noise level can be ignored. In this case the output signal can be represented as the product of two random variables

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